Solid Friction Damping of Mechanical Vibrations

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A theory of solid friction damping of mechanical vibrations is presented that is based on a solid friction mathematical model previously proposed by the author. A summary and improved description of the general analytic features of the solid friction model are given as necessary background for the theory. The Coulomb friction damped oscillator is analyzed to establish an approach to the treatment of a simple friction damped oscillator. The approach then is generalized to treat a more general model of friction where the author's model is used to describe friction force primarily as a function of displacement. The solid friction damped oscillator studied is a wire pendulum where solid friction enters via inelastic flexing of the wire at the support. Theoretical results are generalized to be applicable to other types of oscillators and other sources of solid friction. An expression for the decay rate of the oscillation amplitude envelope of an unforced oscillator is derived. The decay rate and an equivalent linear damping ratio are determined for several values of an exponent parameter in the solid friction model.

I. Introduction

SOME simple tests were run on a wire sample to determine its natural damping characteristics. A weight was attached to the wire and the wire suspended as a simple pendulum under a vacuum. Amplitude of the oscillation was recorded upon release of the pendulum from an initial displacement. Analysis of the amplitude decay envelope data indicated a distinctively nonlinear damping behavior. It was apparent that a nonlinear model for damping was necessary to describe the pendulum damping over a wide range of deflection angles.

To describe the nonlinear behavior, it was thought that the author's model of solid friction, which was devised to describe bearing friction, likely would be applicable in describing the damping process, because the wire damping is the result of hysteretic work losses in the wire as it flexes back and forth in bending, and the solid friction model nicely simulates this type of process.

It was envisioned that it would be possible to investigate and analyze the nonlinear oscillation decay behavior in order to fit parameters of the friction model in the dynamics equation for the wire pendulum such that the observed behavior could be duplicated by simulation.

In the material that follows, a background and essential details of the author's solid friction model are presented. Next, the simple pendulum or spring/mass Coulomb friction damped oscillator is analyzed to show the general analytic approach employed. The general solid friction model then is introduced into the oscillator dynamics equation, and the relation of the damped oscillation amplitude to the solid friction energy dissipation per cycle is obtained. Further analysis yields the energy dissipation per cycle for several solid friction models. The oscillation amplitude decrement per cycle thereby is found as a function of amplitude, making it possible to integrate to obtain the oscillation decay envelope.

II. Background

In an Aerospace Corporation report,[1] a new math model of solid friction was formulated and presented by the author. The model slowly gained acceptance over the years and was used successfully by U.S. Government agencies and their contractors in simulation studies involving primarily ball bearing friction. Some of the studies known to the author have been published in the literature.[2-6] Improvements and extensions of the original model should be credited to Kuo and Singh,[2] Seltzer,[2,6] Nurre,[3] and Osborne and Rittenhouse.[5]

The mathematical model of solid friction that was formulated employs the observation that the time rate of change of solid friction can be expressed as

$$\frac{\mathrm{d}F(x)}{\mathrm{d}t} = \frac{\mathrm{d}F(x)}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}t} \tag{1}$$

where $F(x)$ is a solid friction force which is a function of displacement x only, and which has the characteristic shown in Fig. 1.

The friction force function $F(x)$ monotonically approaches $+F_C$ as long as $\dot{x}$ is positive. When the direction of motion is reversed, $\dot{x}<0$ and $F(x)$ follows the negative of its original shape and approaches $-F_C$. The friction function slope $\mathrm{d}F(x)/\mathrm{d}x$, however, remains positive at all times, even though $\dot{x}$ changes sign. It was discovered that this hysteretic behavior could be simulated with friction slope functions of the form

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = \sigma \left| 1 - \frac{F}{F_c}\,\mathrm{sgn}\,\dot{x} \right|^{i} \mathrm{sgn}\left(1 - \frac{F}{F_c}\,\mathrm{sgn}\,\dot{x}\right) \tag{2}$$

which are illustrated in Fig. 2 for positive velocities. The friction model is seen to behave as a nonlinear "soft" spring with a nearly linear elastic curve for small deflections, which yields and approaches a plastic curve for large deflections.

The quantity σ is the rest stiffness or slope of the force-deflection curve at $F=0$, and F_c is the Coulomb friction force which can also be thought of as a "yield force" or as "running friction force" (for example, as found in bearing friction).

Equations (1) and (2) can be used readily in simulations by employing the conceptual block diagram in Fig. 3. The operation of the friction model simulation loop in Fig. 3 is such that the Coulomb friction force $\pm F_c$ is approached asymptotically and, in principle, the slope function $\mathrm{d}F(x)/\mathrm{d}x$ need not be defined for $|F|>F_c$. However, in actual computer simulation practice, to ensure stability, the friction slope functions should have odd symmetry about the point $F/F_c=1$. This feature is provided by the multiplying factor, sgn $(1-(F/F_c)\,\mathrm{sgn}\,\dot{x})$. A small bias or feedback around the integrator in Fig. 3 also will take care of this stability problem.

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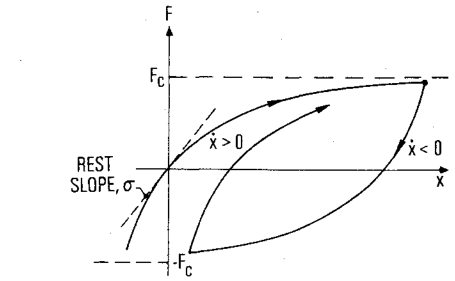

Fig. 1 Typical solid friction force function.

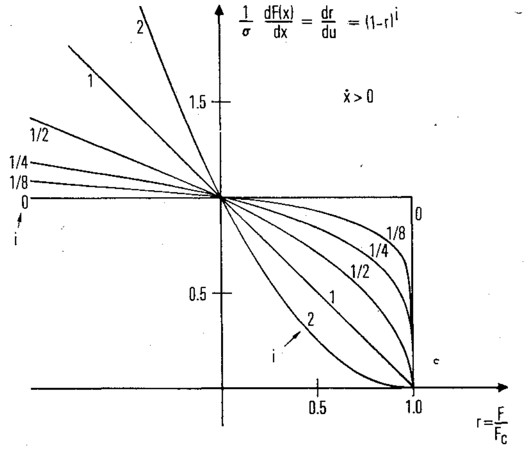

Fig. 2 Friction slope functions.

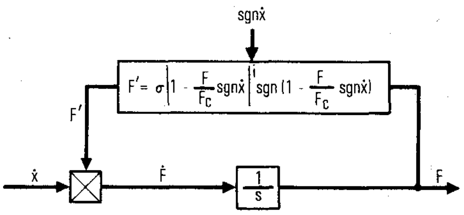

Fig. 3 Solid friction simulator.

The shape of the friction force-deflection functions $F(x)$ for positive velocities can be found by solution of Eq. (2) with $\operatorname{sgn}\dot{x} = +1$. Introducing the dimensionless ratio

$$r = F/F_c \tag{3}$$

and using the initial condition $F(x) = 0$ at $x = 0$ give the result

$$r = 1 - [1 - (1-i)u]^{1/(1-i)} \tag{4}$$

with

$$u < \left(\frac{1}{1-i}\right) \text{ for } i < 1$$

where

$$u = \frac{x}{x_c} \tag{5}$$

is a dimensionless displacement variable, and x_c is a characteristic displacement defined by

$$x_c = \frac{F_c}{\sigma} \tag{6}$$

Several friction force-deflection functions given by Eq. (4) are shown in Fig. 4 for various values of the exponent i. The

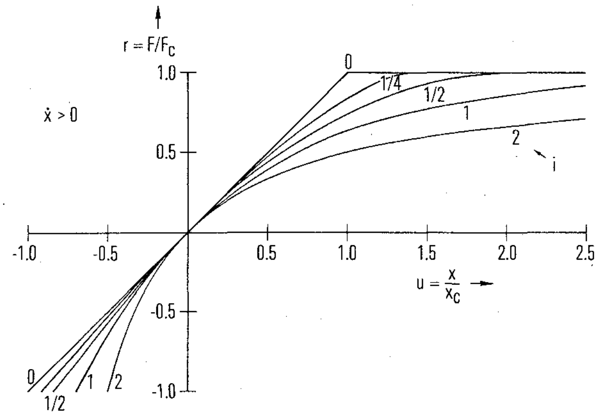

Fig. 4 Force/deflection functions.

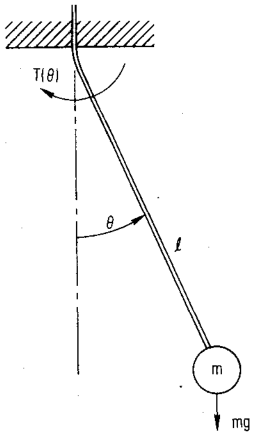

Fig. 5 Wire pendulum.

value of the parameter i defines the basic form of the friction functions and, hence, is used to describe the friction law being used. The functions given by Eqs. (2) and (4) give the same appearance as similar functions found and used by Iwan[7] to model hysteresis damping of structures. Equation (4) is also similar to an equation developed by Ramberg and Osgood[8] used to represent a wide range of stress-strain curves.

The force-deflection curves in Fig. 4 possess the characteristics of ductile type ($i = 1,2$), as well as brittle type ($i = 0$, ¼, ½) materials, and so a suitable solid friction model probably can be fit to the material being studied provided force-deflection test data can be obtained. It was found, however, that a single friction model is not adequate in some cases to simulate accurately the force-deflection data of some physical systems where minor hysteresis loops are considered. One solution to this problem is to employ additional friction models in parallel, with the output friction from each model being added to obtain the total friction. However, as Osborne and Rittenhouse[5] have pointed out, this approach may not be always sufficiently accurate to simulate nonsymmetric or off-axis minor hysteresis loops. Because the hysteresis loops considered in this paper are major loops that are essentially symmetric about the force-deflection origin, the accuracy of the single friction models has been found adequate.

III. The Solid Friction Damped Oscillator

Previous applications of the foregoing solid friction model have been used in studies of control systems where bearing friction was involved.[1-6] These studies showed that the model, when used in computer simulations of system dynamics, gave excellent agreement with test results. Seltzer[6] showed that the model can be used analytically to obtain describing functions for the nonlinear behavior of solid friction, thereby making it possible to predict the existence of and conditions for control system limit cycles due to nonlinear bearing friction. It is suggested that now it will be possible also to predict limit cycles due to nonlinear structural damping because of the identical nature of the two phenomena.

The background theory of the solid friction model has been given, and it now will be applied to the general case of a simple, unforced, second-order oscillator containing solid friction damping. Results should be applicable to the damping of structural oscillations. First, however, the classic Coulomb friction damped oscillator will be discussed to establish an analytic baseline and to fix concepts.

A. Coulomb Friction Damped Oscillator

Consider a mass suspended as a pendulum by a wire as shown in Fig. 5. It is assumed that the wire remains straight, except for local bending at the support point. The equation of motion can be written

$$m\ell\ddot{\theta} = -mg\sin\theta - \frac{T(\theta)}{\ell} \tag{7}$$

where $T(\theta)$ is the opposing torque in the wire at the support, which is generally a nonlinear function of the pendulum deflection angle θ. Assuming θ small and using $\ddot{\theta}=\dot{\theta}(\mathrm{d}\dot{\theta}/\mathrm{d}\theta)$, Eq. (7) can be written as

$$\frac{m\ell^2}{2}\mathrm{d}(\dot{\theta}^2) + \frac{mg\ell}{2}\mathrm{d}(\theta^2) = -T(\theta)\mathrm{d}\theta \tag{8}$$

Let us examine the familiar case of Coulomb friction where, for the pendulum oscillator, the nonlinear torque function is defined by

$$T(\theta) = T_c \operatorname{sgn}\dot{\theta} \tag{9}$$

This form for $T(\theta)$ permits Eq. (8) to be integrated readily over a half cycle, where sgn $\dot{\theta}$ is constant. The result is

$$\dot{\theta}^2 + \frac{g}{\ell}\theta^2 = \frac{-2T_c \operatorname{sgn}\dot{\theta}}{m\ell^2}\theta + C_I \tag{10}$$

where C_I is a constant of integration. Recognizing that $\dot{\theta}=0$ at the beginning of a half cycle allows Eq. (10) to be put in the form

$$\dot{\theta}^2 + (g/\ell)(\theta+\theta_c \operatorname{sgn}\dot{\theta})^2 = R^2 \tag{11}$$

where

$$R^2 = (g/\ell)(\theta_I - \theta_c)^2 \tag{12}$$

θ_I = magnitude of displacement at beginning of half cycle, and

$$\theta_c = T_c/mg\ell \tag{13}$$

The amplitude decay rate or the change in amplitude per cycle is

$$\mathrm{d}\bar{\theta}/\mathrm{d}N = -2\Delta\theta \tag{14}$$

where N is the cycle number, $\bar{\theta}$ is the amplitude at the beginning of the cycle, and $\Delta\theta$ is the decrease in amplitude in one-half cycle, which is deduced to be

$$\Delta\theta = 2\theta_c \tag{15}$$

It is noted that the energy dissipated per half cycle, ΔE, due to the Coulomb friction, is

$$\Delta E = 2T_c\bar{\theta} \tag{16}$$

We now introduce the spring rate k_n of an equivalent spring mass oscillator

$$k_n = mg\ell \tag{17}$$

and employ Eqs. (13), (15), and (16) in (14) to obtain

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}N} = \frac{-4T_c}{k_n} \tag{18}$$

or

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}N} = \frac{-2\Delta E}{k_n\bar{\theta}} \tag{19}$$

It will be seen that Eq. (16) is a limiting form of a more general relation for energy dissipated per half cycle, and that Eq. (19) is the relation for the amplitude decay rate of solid friction damped systems.

Equation (18) can be integrated directly to obtain the decay envelope relation

$$(\theta - \theta_0) = \frac{-4T_c}{kn}N \tag{20}$$

where θ_0 is the initial amplitude at $N=0$. Equation (20) shows the linear nature of the oscillation decay envelope of classical Coulomb friction for $\theta > \theta_c$.

B. General Solid Friction Damped Oscillator

Let us proceed again from Eq. (7) and rephrase this solid friction damped oscillator equation by using the defining Eq. (17) for k_n. Equation (7) becomes, employing the approximation $\sin\theta \approx \theta$

$$m\ell^2\ddot{\theta} = -k_n\theta - T(\theta) \tag{21}$$

We use $\ddot{\theta}=\dot{\theta}\,\mathrm{d}\dot{\theta}/\mathrm{d}\theta$ in Eq. (21) and integrate to find

$$m\ell^2\int_{\dot{\theta}_I}^{\dot{\theta}_2}\dot{\theta}\mathrm{d}\dot{\theta} + \int_{\theta_I}^{\theta_2}[k_n\theta + T(\theta)]\mathrm{d}\theta = 0 \tag{22}$$

or

$$\frac{m\ell^2}{2}(\dot{\theta}_2^2 - \dot{\theta}_I^2) + \frac{k_n}{2}(\theta_2^2 - \theta_I^2) = -\int_{\theta_I}^{\theta_2} T(\theta)\mathrm{d}\theta \tag{23}$$

If we choose the limits of integration to be the extremes of a half cycle of motion, then $\dot{\theta}_2$ and $\dot{\theta}_1$ are zero and the first term in Eqs. (22) and (23) will be zero. Under these conditions, Eq. (22) becomes

$$\int_{\theta_I}^{\theta_2}[k_n\theta + T(\theta)]\mathrm{d}\theta = 0 \tag{24}$$

1. Nonlinear "Spring" Only

Let us now examine the nature of this integral for the case where the oscillator does not possess a linear spring, i.e., $k_n = 0$. We then have the nonlinear "spring" possessing damping characteristics,

$$\int_{\theta_I}^{\theta_2} T(\theta)\mathrm{d}\theta = 0 \tag{25}$$

or, in terms of the nondimensional variables r and u,

$$\int_{u_I}^{u_2} r(u)\ \mathrm{d}u = 0 \tag{26}$$

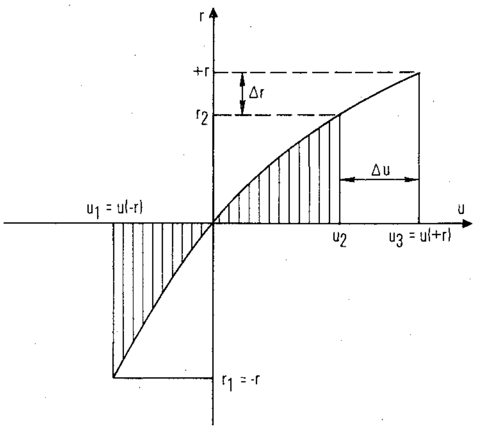

Fig. 6 Amplitude decrement in a half cycle.

This relation is illustrated in Fig. 6, which shows the integral between the limits u_1 and u_2 to be zero, i.e., the shaded areas are equal. The torque-deflection curve from $u_1 = u(-r)$ is shown extended past u_2 to the point $u_3 = u(+r)$. It is seen from the figure that

$$\int_{u_1}^{u_2} r\mathrm{d}u = \int_{u_1}^{u_3} r\mathrm{d}u - \int_{u_2}^{u_3} r\mathrm{d}u \tag{27}$$

and that

$$\int_{u_2}^{u_3} r\mathrm{d}u \approx \left(r - \frac{\Delta r}{2}\right)\Delta u \approx r\Delta u \tag{28}$$

where Δr is assumed to be small compared to r. Substituting Eqs. (26) and (28) into (27) then results in

$$r\Delta u = \int_{u_1}^{u_3} r\mathrm{d}u \tag{29}$$

The quantity Δu is the decrement in the amplitude of oscillation per half cycle. This interpretation of Δu is explained as follows.

Suppose that a small amount of energy were added each half cycle, such that the magnitudes of the maximum restoring torque at the extremes of a half cycle of motion were equal. Under this condition, the peak-to-peak amplitude of the oscillatory motion would be sustained and would neither increase nor decrease. Thus, it can be seen that an oscillation with no decay would swing over a range from u_1 to u_3. It follows, therefore, that Δu represents the loss in amplitude over a half cycle.

We now define the amplitude $\bar{u}$

$$\bar{u} = \frac{u(+r) - u(-r)}{2} \tag{30}$$

which is seen to be the amplitude of the damped oscillator at the beginning of the half cycle. The amplitude at the end of the half cycle is obviously

$$\bar{u}_{j+1} = \bar{u}_j - \Delta u \tag{31}$$

where j denotes the half cycle number.

It becomes apparent that the amount of energy that must be added to sustain the amplitude of oscillation is the area under the torque deflection curve from u_2 to u_3, and that this must be equal to ΔE, the energy dissipated over the half cycle due to solid friction. We therefore define ΔE generally as the quantity on the right-hand side of Eq. (29). After denormalizing,

$$\Delta E = T_c\theta_c \int_{u_1}^{u_3} r\mathrm{d}u \tag{32}$$

2. Linear and Nonlinear Springs in Parallel

It is now possible to apply this approach to the case where a linear spring is in parallel with the nonlinear spring such that Eq. (24) applies. In terms of the nondimensional variables, Eq. (24) becomes

$$\int_{u_1}^{u_2} \left(\frac{k_n}{\sigma} u + r\right) \mathrm{d}u = 0 \tag{33}$$

Referring to Fig. 7 for this case and following the same procedure as before, we have

$$0 = \int_{u_1}^{u_3} \frac{k_n}{\sigma} u\mathrm{d}u - \int_{u_2}^{u_3} \frac{k_n}{\sigma} u\mathrm{d}u + \int_{u_1}^{u_3} r\mathrm{d}u - \int_{u_2}^{u_3} r\mathrm{d}u \tag{34}$$

The first term in Eq. (34) is zero, the second term can be approximated by $(k_n/\sigma)\bar{u}\Delta u$, and the remaining terms have been found previously, which leads us to a relation similar in form to Eq. (29)

$$\left(r(\bar{u}) + \frac{k_n}{\sigma}\bar{u}\right)\Delta u = \int_{u_1}^{u_3} r\mathrm{d}u \tag{35}$$

Denormalizing Eq. (35) and using Eq. (32) gives

$$\Delta\theta = \frac{\Delta E}{T(\bar{\theta}) + k_n\bar{\theta}} \tag{36}$$

where

$$\bar{\theta} = \theta_c \bar{u} \tag{37}$$

Using Eq. (14) in (36) gives

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}N} = \frac{-2\Delta E}{T(\bar{\theta}) + k_n\bar{\theta}} \tag{38}$$

which corresponds to Eq. (19) for simple Coulomb friction if $T(\bar{\theta}) << k_n\bar{\theta}$. At large amplitudes, the nonlinear friction torque is saturated such that $T(\bar{\theta}) \approx T_c$ will usually be small compared to $k_n\bar{\theta}$. At low amplitudes, the wire is nearly elastic, and $T(\bar{\theta}) \approx \sigma\theta$ and Eq. (38) becomes

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}N} = -\frac{2\Delta E}{(\sigma + k_n)\bar{\theta}} \tag{39}$$

indicating as may be expected that at low amplitudes, the nonlinear friction element behaves as a spring where its rest stiffness σ adds in parallel with the linear stiffness k_n.

Equation (36) for the amplitude decay is viewed as being basic for solid friction damped oscillators and in conjunction with the difference equation given by Eq. (31) permits determination of the oscillation decay envelope. Alternatively, solution of the differential equation given by Eq. (38) also will give the oscillation decay envelope as a function of N. Equation (38) also can be used as a basis for comparison with the damping characteristics of a linear viscous damped oscillator which has an amplitude decay rate given by

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}N} = -2\pi\zeta\bar{\theta} \tag{40}$$

where ζ is an equivalent linear damping ratio. Then, comparing Eq. (38) with (40), we deduce that the equivalent linear damping ratio can be expressed as

$$\zeta = \frac{\Delta E}{\pi\bar{\theta}(k_n\bar{\theta} + T(\bar{\theta}))} \tag{41}$$

IV. Solid Friction Damping

The foregoing examination of the general case of damping of an oscillator via the mechanism of solid friction has indicated that the amplitude decay rate, the amplitude decay envelope, and an equivalent linear damping ratio can be determined if analytical expressions can be found for the solid friction energy dissipation. We shall proceed to do this using force-deflection functions expressed by Eq. (4).

A. Solid Friction Energy Dissipation

The energy loss due to hysteretic solid friction damping over a half cycle has been defined by Eq. (32). It also can be expressed as

$$\frac{\Delta E}{2T_c\theta_c} = \frac{1}{2}\int_{u(-r)}^{u(+r)} r\,\mathrm{d}u \tag{42}$$

Alternatively, the integration in Eq. (42) can be carried out with respect to torque rather than displacement using the expression

$$\Delta E = \int_{-T}^{T} \frac{T}{\mathrm{d}T/\mathrm{d}\theta}\,\mathrm{d}T \tag{43}$$

or, in terms of the dimensionless variables r and u,

$$\frac{\Delta E(r)}{2T_c\theta_c} = \frac{1}{2}\int_{-r}^{r} \frac{r}{(\mathrm{d}r/\mathrm{d}u)(A)}\,\mathrm{d}r \tag{44}$$

After integration, ΔE can be found as a function of u, the displacement variable, by substitution of $r=r(u)$ from Eq. (4) into Eq. (44). However, the amplitude at the beginning of a half cycle is not the same as u, but is equal to $\bar{u}$ given by Eq. (30). We therefore evaluate $r=r(\bar{u})$ and substitute into the solution of Eq. (44) to find $\Delta E(\bar{u})/2T_c\theta_c$. The results for $i=0, 1, 2$ are

$$\frac{\Delta E}{2T_c\theta_c}(\bar{u}) = \left\{\begin{array}{ll} \left(\begin{array}{l} 0, \bar{u}<1 \\ (\bar{u}-1), \bar{u}>1 \end{array}\right), & i=0 \\ \bar{u}-\tanh \bar{u}, & i=1 \\ \bar{u}-\tfrac{1}{2}\ln\left[\dfrac{1+r(\bar{u})}{1-r(\bar{u})}\right], & i=2 \end{array}\right\} \tag{45}$$

It is seen that the $\Delta E(\bar{u})/2T_c\theta_c$ functions for various values of i reveal that they have a general form involving two terms, the first of which is $\bar{u}$. The second term becomes small compared to $\bar{u}$ as $\bar{u}$ gets large. For $\bar{u}>>1$, it is found that $\Delta E(\bar{u})/2T_c\theta_c$ may be approximated by $\bar{u}$ or that

$$\Delta E(\bar{\theta}) \approx 2T_c\bar{\theta}, \quad \bar{\theta}>>\theta_c \tag{46}$$

which is identical to Eq. (16) found for the simple Coulomb models.

An important approximation for the energy dissipation at low amplitudes is found by assuming $r<<1$ in the $\Delta E(r)/2T_c\theta_c$ expression for general i values. Using the first terms in the binomial expansions up to the r^3 term for the $(1\pm r)$ quantities [in the $\Delta E(r)/2T_c\theta_c$ expression] and noting that $r\approx\bar{u}$ for $r<<1$, it is found that

$$\frac{\Delta E(\bar{u})}{2T_c\theta_c} = \frac{i}{3}\bar{u}^3, \quad \bar{u}<<1 \tag{47}$$

This approximation is valid for all values of i investigated, including $i=1$. It indicates that the mechanism of solid friction damping results from curvature in the force-deflection curve manifested by third-order terms in the power series expansions of the functions at low amplitudes. It is noted that for $i=0$, no energy dissipation exists at low amplitudes, i.e., the force-deflection curve is purely elastic for $\bar{u}<1$. For this reason, the uniform friction model ($i=0$) is viewed as being a degenerate case and of little value in simulating continuous solid friction damping.

It also is observed that the energy dissipation is proportional to i as indicated in Eq. (47). In addition, it can be shown that the slope of the friction slope functions, given by Eq. (2) and shown in Fig. 2, for $r<<1$ is equal to i. These observations help give some insight to the source of solid friction damping and the characterization of damping by means of the friction model and its parameters.

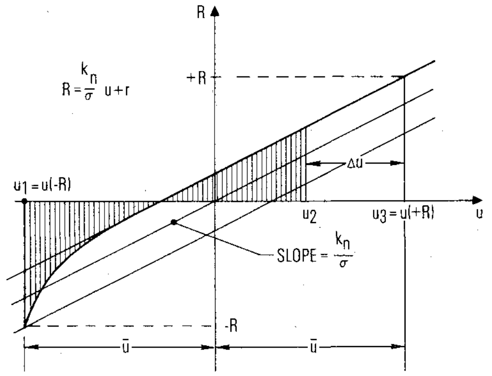

Fig. 7 Energy loss and amplitude decrement per half cycle—nonlinear plus linear spring.

B. Amplitude Decay Rate

The amplitude decay rate, given by Eq. (38), can be expressed as

$$-\frac{1}{4}\left[\frac{r(\bar{u})}{\bar{u}}+\frac{k_n}{\sigma}\right]\frac{\mathrm{d}\bar{u}}{\mathrm{d}N} = \frac{1}{\bar{u}}\cdot\frac{\Delta E(\bar{u})}{2T_c\theta_c} \tag{48}$$

The right side of Eq. (48) is the energy dissipation function $\Delta E(\bar{u})/2T_c\theta_c$, divided by $\bar{u}$. Plots of decay rate functions given by Eq. (45) are presented in Fig. 8.

The linear viscous damped oscillator amplitude decay rate given by Eq. (40) can be expressed as

$$\frac{0.25}{2\pi\zeta}\frac{\mathrm{d}\bar{u}}{\mathrm{d}N} = -0.25\bar{u} \tag{49}$$

in order that the solid friction amplitude decay rates can be compared to that of viscous friction. The factor 0.25 has been

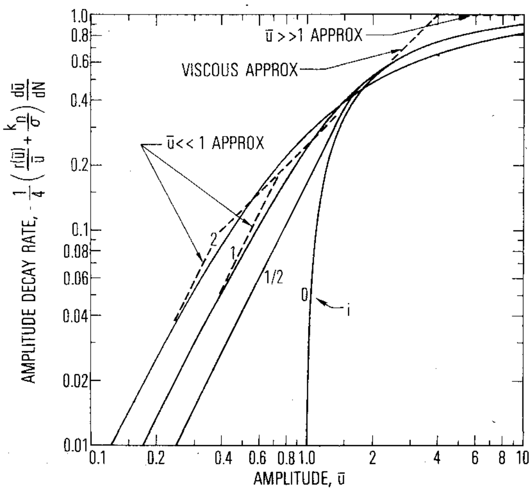

Fig. 8 Amplitude decay rate.

introduced on both sides of Eq. (49) in order to obtain a reasonably good empirical fit of the right side of Eq. (49) with the plotted curves of the right side of Eq. (48).

Figure 8 shows fairly clearly that the curves can be separated into three regions, described as follows

1) Coulomb region

$$\frac{-k_n}{4\sigma}\frac{\mathrm{d}\bar{u}}{\mathrm{d}N}\approx 1,\quad \bar{u}>>1 \tag{50}$$

and damping behavior is that of Coulomb friction.

2) Viscous region

$$-\frac{1}{4}\left(\frac{r(\bar{u})}{\bar{u}}+\frac{k_n}{\sigma}\right)\frac{\mathrm{d}\bar{u}}{\mathrm{d}N}\approx 0.25\bar{u},\quad \bar{u}\approx 1 \tag{51}$$

where the factor 0.25 is empirical and damping behavior is that of viscous friction.

3) Structural region

$$-\frac{1}{4}\left(1+\frac{k_n}{\sigma}\right)\frac{\mathrm{d}\bar{u}}{\mathrm{d}N}\approx\frac{i\bar{u}^2}{3},\quad \bar{u}<<1 \tag{52}$$

and damping behavior is similar to that found in structural damping.

C. Equivalent Linear Damping Ratio

Although the amplitude decay rate functions should describe solid friction damping adequately, it is desirable to describe damping in terms of the equivalent linear damping ratio defined by Eq. (41), which can be expressed in terms of $\bar{u}$

$$\left(\frac{r(\bar{u})}{\bar{u}}+\frac{k_n}{\sigma}\right)\zeta=\frac{2}{\pi\bar{u}^2}\;\frac{\Delta E(\bar{u})}{2T_c\theta_c} \tag{53}$$

Equation (53) is plotted in Fig. 9. The three regions alluded to are again evident

1) Coulomb region

$$\frac{k_n}{\sigma}-\zeta\approx\frac{2}{\pi\bar{u}},\quad \bar{u}>>1 \tag{54}$$

2) Viscous region

$$\left(\frac{r(\bar{u})}{\bar{u}}+\frac{kn}{\sigma}\right)\zeta\approx\frac{1}{2\pi},\quad \bar{u}\approx 1 \tag{55}$$

3) Structural region

$$\left(1+\frac{k_n}{\sigma}\right)\zeta\approx\frac{2i}{3\pi}\bar{u},\quad \bar{u}<<1 \tag{56}$$

The first relationship implies that $r/\bar{u}$ is small compared to k_n/σ at large amplitudes, and that the equivalent linear damping ratio is inversely proportional to amplitude. If $k_n=0$, this approximation obviously is not valid, and the $r/\bar{u}$ term must be used; however, the initial decay in this case will be extremely rapid, and a Coulomb damping phase will not occur. The second relation reveals that solid friction damping appears to be the same as viscous friction when the amplitude is near the value of the characteristic displacement. The last relationship indicates that the equivalent linear damping ratio is proportional to the amplitude of the oscillation $\bar{u}$ at low amplitudes. This is a well-known behaviorial characteristic of structural damping; i.e., the damping ratio or damping factor becomes vanishingly small as the amplitude of oscillation decreases. This characteristic has, in the past, been the basis for assuming or choosing extremely small values of the equivalent linear damping ratio for criteria in the design of high-performance servos and structures. Where an engineering design becomes critically dependent upon knowledge of damping behavior, a nonlinear friction model of the type presented here may be applied fruitfully to represent accurately the physics of structural damping without resorting to restrictively small values of equivalent linear damping ratio.

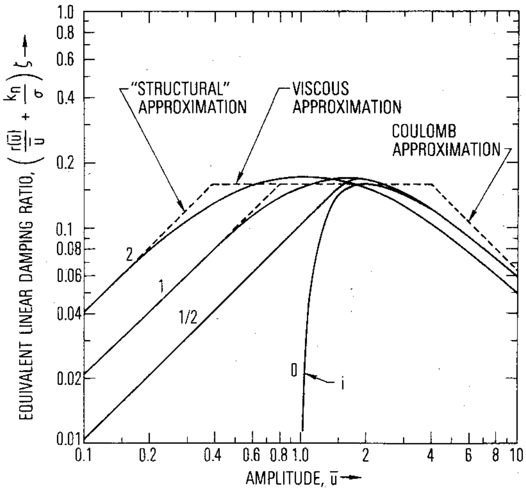

Fig. 9 Equivalent linear damping ratio.

V. Oscillation Decay Envelope

The oscillation decay envelopes for the various solid friction models can be determined using the difference Eq. (31) and the expression for the amplitude decrease per half cycle given by Eq. (36), or by solving the differential Eq. (38). It was found that both approaches give the same result under conditions where the amplitude change per cycle is small. These conditions exist when the linear spring rate is large compared to the nonlinear spring rate, i.e., $k_n/\sigma>>1$, which was the case for the wire pendulum that was tested. For this reason, the particular case where $k_n/\sigma>>1$ was considered and amplitude envelopes were obtained. In addition, the case where the linear spring is absent also was investigated.

For the first case, $r(\bar{u})/\bar{u}$ was assumed small compared to k_n/σ everywhere, and Eqs. (36) and (31) were put in the approximation form

$$\Delta u=\frac{\sigma}{k_n}\frac{2[\Delta E(\bar{u})/2T_c\theta_c]}{\bar{u}} \tag{57}$$

$$\bar{u}_{j+1}=\bar{u}_j-\Delta u \tag{58}$$

and the differential Eq. (39) was put in the approximation form

$$\frac{k_n}{\sigma}\frac{\mathrm{d}\bar{u}}{\mathrm{d}N}=\frac{-2\Delta E(\bar{u})/T_c\theta_c}{\bar{u}} \tag{59}$$

By letting $k_n/\sigma=4$ in Eq. (57) and using the energy dissipation function for $i=1$, Eqs. (57) and (58) were solved numerically. The solution is shown in Fig. 10, where $\bar{u}$ is plotted vs $(\sigma/k_n)N$. The initial amplitude $\bar{u}=10$ was chosen arbitrarily. This solution checks closely with the solution obtained for the differential Eq. (59). However, if k_n/σ is set equal to 1.0, the difference equation solution deviates from the differential equation solution excessively at the larger amplitudes (i.e., $\bar{u}\gtrsim 1$) due to approximations made in Eq. (28). The differential equation therefore is regarded as providing a better description of the continuous decay envelope, although the difference equation realistically describes the decay as a discrete process.

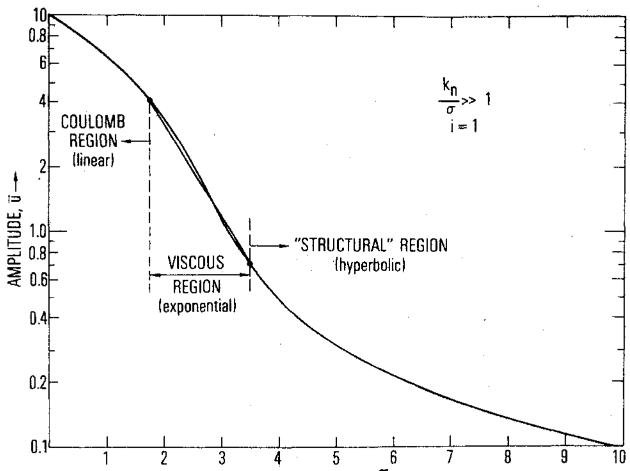

Fig. 10 Oscillator decay envelope.

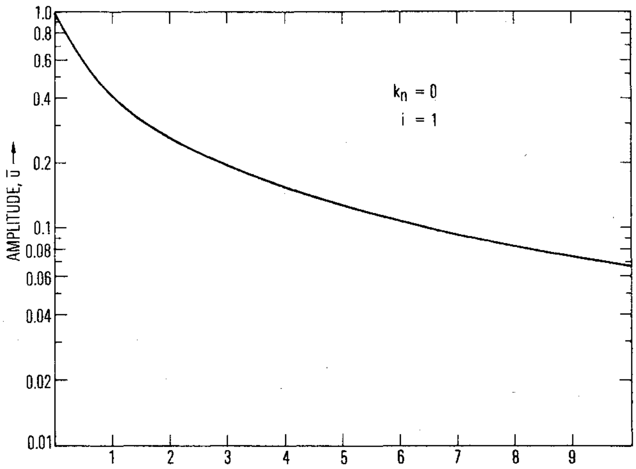

Fig. 11 "Structural" decay envelope.

Figure 10 shows the three regions described previously. If solutions to Eqs. (50-52) are obtained with the initial condition $u=u_0$ at $N=0$ under the assumption that $k_n/\sigma >> 1$, the three amplitude decay regions in Fig. 10 can be described as follows

a) Coulomb region

$$(\bar{u}-\bar{u}_0) = -4(\sigma/k_n)N, \quad \bar{u} \gtrsim 4 \tag{60}$$

and the envelope decay is linear with N.

b) Viscous region

$$\bar{u}=\bar{u}_0\exp\left[-\left(\frac{\sigma}{k_n}N\right)\right], \quad 4 \gtrsim \bar{u} \gtrsim 0.7 \tag{61}$$

and the envelope decay is exponential with N.

c) Structural region

$$\bar{u}=\left(\frac{4i}{3}\frac{\sigma}{k_n}N+\frac{1}{\bar{u}_0}\right)^{-1}, \quad \bar{u} \lesssim 0.7 \tag{62}$$

and the envelope decay is hyperbolic with N.

In the Coulomb region, the decay envelope is described by Eq. (60), which corresponds to the linear decay envelope of classical Coulomb friction given by Eq. (20).

In the viscous region in the semilog plot of Fig. 10, the exponential decay envelope is approximately a straight line from $\bar{u} \approx 4$ to $\bar{u} \approx 0.7$. The constant slope line on the figure in this range has a slope that corresponds to the linear equivalent damping ratio empirical value of $0.16 \approx 1/2\pi$. This is indicated in Fig. 9 as the constant value of the approximate equivalent linear damping ratio in the same region.

In the structural region, the amplitude $\bar{u}$ varies inversely with N and, hence, the decay envelope is described as hyperbolic and approaches zero asymptotically. The time required to reduce amplitude by a given percentage can be shown from Eq. (62) to be proportional to amplitude. This is a way of saying that the equivalent linear damping ratio is proportional to the amplitude. This is not to say, however, that the solid friction damping mechanism ceases at low levels. In fact, it seems important to draw the conclusion that the classical concept of viscous damping should be set aside at low amplitudes, and the decay envelope should be viewed simply as being hyperbolic rather than exponential.

For the second case where the linear spring is absent, $k_n=0$ and the differential Eq. (38) can be put in the approximation form

$$\frac{d\bar{u}}{dN} \approx \frac{2\Delta E(\bar{u})/T_c\theta_c}{r(\bar{u})} \tag{63}$$

A computer solution of Eq. (63) was obtained and is shown in Fig. 11. The plot of $\bar{u}$ vs N starts at $\bar{u}=1$ rather than $\bar{u}=10$ as in Fig. 10, because the extremely high decay rate values for $\bar{u}>1$ indicate that the motion will be damped to $\bar{u}<1$ in the first half cycle or so. It is observed, however, that the decay envelope in Fig. 11 is almost identical to that of Fig. 10 for $\bar{u} \lesssim 0.7$, which is the "structural region." The case where $k_n=0$ therefore is seen to apply to structural materials; hence, the name given this region is appropriate.

The general case where both terms $r/\bar{u}$ and k_n/σ are considered is not treated here. However, the character of the decay envelopes for the general case will be similar to those analyzed.

VI. Conclusions

The solid friction models behave like Coulomb friction at large amplitudes of oscillation, and the amplitude decay envelopes decrease linearly with the number of cycles or with time. The behavior is similar to viscous friction when the amplitude of oscillation is near the characteristic displacement x_c and the amplitude decay envelopes decrease exponentially with the number of cycles.

At low amplitudes, the solid friction models behave in a way that is described as "structural" damping, wherein the equivalent linear damping ratio varies directly with the amplitude of oscillation. In the low-amplitude range, the amplitude decay envelopes decrease hyperbolically with the number of cycles or with time.

The "structural" equivalent damping ratio is proportional to the exponent i in the solid friction model which characterizes the curvature on the load-deflection curve of the solid friction material.

Computer simulations using the solid friction models will, of course, obviate the need for extensive analysis in most problems where the solid friction models are applied. Simulation is certainly the approach to take in the study of large, high-order state equation systems. The usefulness of the second-order nonlinear oscillator analysis presented here lies in gaining insight and understanding of friction damping behavior.

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